

HECKE-TYPE CONGRUENCES FOR TWO SMALLEST PARTS FUNCTIONS

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ABSTRACT. We prove infinitely many congruences modulo 3, 5, and powers of 2 for the overpartition function $\overline{p}(n)$ and two smallest parts functions: $\overline{\text{spt}}\overline{1}(n)$ for overpartitions and $\text{M2spt}(n)$ for partitions without repeated odd parts. These resemble the Hecke-type congruences found by Atkin for the partition function $p(n)$ in 1966 and Garvan for the smallest parts function $\text{spt}(n)$ in 2010. The proofs depend on congruences between the generating functions for $\overline{p}(n)$, $\overline{\text{spt}}\overline{1}(n)$, and $\text{M2spt}(n)$ and eigenforms for the half-integral weight Hecke operator $T(\ell^2)$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\text{spt}(n)$ denote the number of smallest parts in the partitions of n . For example, there are 7 partitions of 5,

$$5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1$$

so $\text{spt}(5) = 14$. The spt function was introduced by Andrews [3] and has since been studied extensively (see, for instance, [1, 8, 9, 10, 11, 16]). In particular, Garvan gave the following congruences for the spt function in [10].

Theorem 1.1. *Let $s_\ell = (\ell^2 - 1)/24$. Then the following are true:*

(i) *If $\ell > 5$ is prime then for any $n \geq 1$, we have*

$$\text{spt}(\ell^2 n - s_\ell) + \left(\frac{12}{\ell}\right) \left(\frac{1 - 24n}{\ell}\right) \text{spt}(n) + \ell \text{spt}\left(\frac{n + s_\ell}{\ell^2}\right) \equiv \left(\frac{12}{\ell}\right) (1 + \ell) \text{spt}(n) \pmod{72}. \quad (1.1)$$

(ii) *If $\ell \geq 5$ is prime, $t = 5, 7$ or 13 and $\ell \neq t$ then for any $n \geq 1$, we have*

$$\text{spt}(\ell^2 n - s_\ell) + \left(\frac{12}{\ell}\right) \left(\frac{1 - 24n}{\ell}\right) \text{spt}(n) + \ell \text{spt}\left(\frac{n + s_\ell}{\ell^2}\right) \equiv \left(\frac{12}{\ell}\right) (1 + \ell) \text{spt}(n) \pmod{t}. \quad (1.2)$$

In this paper we prove results similar to (1.1) and (1.2) for the overpartition function $\overline{p}(n)$ and for two other smallest parts functions, $\overline{\text{spt}}\overline{1}(n)$ and $\text{M2spt}(n)$.

An overpartition is a partition in which the first occurrence of each distinct part may be overlined or not. For example, there are 8 overpartitions of 3,

$$3, \bar{3}, 2 + 1, \bar{2} + 1, 2 + \bar{1}, \bar{2} + \bar{1}, 1 + 1 + 1, \bar{1} + 1 + 1,$$

so $\overline{p}(3) = 8$. The generating function for overpartitions is

$$\overline{P}(\tau) = \sum_{n=0}^{\infty} \overline{p}(n) q^n := \frac{\eta(2\tau)}{\eta^2(\tau)} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + O(q^6),$$

where $\eta(\tau)$ is the Dedekind η -function $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ and $q := \exp(2\pi i \tau)$. Define $\overline{\text{spt}}\overline{1}(n)$ to be the number of smallest parts in the overpartitions of n having odd smallest part. For example, the overpartitions of 4 having odd smallest part are

$$3 + 1, \bar{3} + 1, 3 + \bar{1}, \bar{3} + \bar{1}, 2 + 1 + 1, \bar{2} + 1 + 1, 2 + \bar{1} + 1, \bar{2} + \bar{1} + 1, 1 + 1 + 1 + 1, \bar{1} + 1 + 1 + 1,$$

so $\overline{\text{spt}}\overline{1}(4) = 20$. The function $\overline{\text{spt}}\overline{1}(n)$ has been studied by many authors including Bringmann, Lovejoy and Osburn [5], who proved the congruence

$$\overline{\text{spt}}\overline{1}(\ell^2 n) + \left(\frac{-n}{\ell}\right) \overline{\text{spt}}\overline{1}(n) + \ell \overline{\text{spt}}\overline{1}(n/\ell^2) \equiv (1 + \ell) \overline{\text{spt}}\overline{1}(n) \pmod{3}. \quad (1.3)$$

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In Theorem 1.2 we recover (1.3) and extend the modulus to $2^8 \cdot 3 \cdot 5$.

By [5, Section 7], the generating function for $\overline{\text{spt1}}(n)$ is

$$\begin{aligned} \overline{S}(\tau) &= \sum_{n=1}^{\infty} \overline{\text{spt1}}(n) q^n = \left(\prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} \right) \left(\sum_{n=1}^{\infty} \frac{2nq^n}{1-q^{2n}} + \sum_{n \neq 0} \frac{4(-1)^n q^{n^2+n}(1+q^{2n}+q^{3n})}{(1-q^{2n})(1-q^{4n})} \right) \\ &= 2q + 4q^2 + 12q^3 + 20q^4 + 40q^5 + O(q^6), \end{aligned}$$

and we define the function \overline{M} by

$$\overline{M}(\tau) := \overline{S}(\tau) + \frac{1}{12} \overline{P}(\tau)(E_2(\tau) - 4E_2(2\tau)), \quad (1.4)$$

where E_2 is the weight 2 quasimodular Eisenstein series

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

The function \overline{M} is a weight $3/2$ mock modular form. Ahlgren, Bringmann, and Lovejoy [1] showed that $\overline{M}(\tau)$ is an eigenform for the weight $3/2$ Hecke operator $T_{\frac{3}{2}}(\ell^2)$, and we use this fact to prove the following theorem.

Theorem 1.2. *Let ℓ be an odd prime, and define*

$$\alpha := \begin{cases} 6 & \text{if } \ell \equiv 3 \pmod{8}, \\ 7 & \text{if } \ell \equiv 5, 7 \pmod{8}, \\ 8 & \text{if } \ell \equiv 1 \pmod{8}. \end{cases}$$

Then for $t \in \{2^\alpha, 3, 5\}$, $\ell \neq t$, and $n \geq 1$, we have

$$\overline{\text{spt1}}(\ell^2 n) + \left(\frac{-n}{\ell} \right) \overline{\text{spt1}}(n) + \ell \overline{\text{spt1}}(n/\ell^2) \equiv (1 + \ell) \overline{\text{spt1}}(n) \pmod{t}. \quad (1.5)$$

Next, define $\text{M2spt}(n)$ to be the restriction of $\text{spt}(n)$ to those partitions without repeated odd parts and whose smallest part is even. For example, $\text{M2spt}(7) = 3$ since the partitions of 7 without repeated odd parts are (with even smallest parts circled)

$$7, 6 + 1, 5 + \textcircled{2}, 4 + 3, 4 + 2 + 1, 3 + \textcircled{2} + \textcircled{2}, 2 + 2 + 2 + 1.$$

By [5, Section 7], the generating function for $\text{M2spt}(n)$ is

$$\begin{aligned} \sum_{n=1}^{\infty} \text{M2spt}(n) q^n &= \left(\prod_{n=1}^{\infty} \frac{1+q^{2n-1}}{1-q^{2n}} \right) \left(\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} + \sum_{n \neq 0} \frac{(-1)^n q^{2n^2+n}}{(1-q^{2n})^2} \right) \\ &= q^2 + 3q^4 + q^5 + 5q^6 + 3q^7 + O(q^8), \end{aligned}$$

and we define functions $S2$ and $M2$ by

$$S2(\tau) := \sum_{n=0}^{\infty} (-1)^n \text{M2spt}(n) q^{8n-1}, \quad (1.6)$$

$$M2(\tau) := S2(\tau) + \frac{1}{24} R(\tau)(E_2(16\tau) - E_2(8\tau)), \quad (1.7)$$

where $R(\tau)$ is the generating function for partitions without repeated odd parts, given by

$$R(\tau) = \sum_{n=0}^{\infty} \text{m2}(n) q^{8n-1} := \frac{\eta(8\tau)}{\eta^2(16\tau)} = \frac{1}{q} - q^7 + q^{15} - 2q^{23} + 3q^{31} + O(q^{32}).$$

The situation here is similar to that of $\overline{\text{spt1}}(n)$. The function $M2$ is also a weight $3/2$ mock modular form, and the generating functions $\overline{P}(\tau)$ and $R(\tau)$ are related under the Fricke involution W_{16} . Ahlgren, Bringmann, and Lovejoy [1] proved that $M2(\tau)$ is an eigenform for the weight $3/2$ Hecke operator $T_{\frac{3}{2}}(\ell^2)$, and we use this to prove the following result.

Theorem 1.3. *Let ℓ be an odd prime, and define*

$$s_\ell := (\ell^2 - 1)/8 \quad \text{and} \quad \beta := \begin{cases} 1 & \text{if } \ell \equiv 3 \pmod{8}, \\ 2 & \text{if } \ell \equiv 5 \pmod{8}, \\ 3 & \text{if } \ell \equiv 1, 7 \pmod{8}. \end{cases}$$

Then for $t \in \{2^\beta, 3, 5\}$, $\ell \neq t$ and $n \geq 1$, we have

$$\text{M2spt}(\ell^2 n - s_\ell) + \left(\frac{2}{\ell}\right) \left(\frac{1-8n}{\ell}\right) \text{M2spt}(n) + \ell \text{M2spt}\left(\frac{n+s_\ell}{\ell^2}\right) \equiv \left(\frac{2}{\ell}\right) (1+\ell) \text{M2spt}(n) \pmod{t}. \quad (1.8)$$

Lastly, we give Hecke-type congruences for overpartitions modulo powers of 2. Many authors have found congruences for $\bar{p}(n)$ modulo powers of 2 (see, for example [7, 12, 14, 13]), such as

$$\bar{p}(8n+7) \equiv 0 \pmod{64}.$$

In this paper, we employ the fact that $\bar{P}(\tau)$ and $q \frac{d}{dq} \bar{P}(\tau)$ are congruent to eigenforms for the Hecke operators $T(\ell^2)$ modulo powers of 2 to prove the following theorem.

Theorem 1.4. *Let ℓ be an odd prime, and define*

$$\gamma := \begin{cases} 5 & \text{if } \ell \equiv 3 \pmod{8}, \\ 6 & \text{if } \ell \equiv 5, 7 \pmod{8}, \\ 7 & \text{if } \ell \equiv 1 \pmod{8}. \end{cases}$$

Then for $n \geq 0$ we have

$$\bar{p}(\ell^2 n) + \left(\frac{-n}{\ell}\right) \ell^{-2} \bar{p}(n) + \ell^{-3} \bar{p}(n/\ell^2) \equiv (1+\ell) \bar{p}(n) \pmod{16}, \quad (1.9)$$

$$\ell^2 n \bar{p}(\ell^2 n) + \left(\frac{-n}{\ell}\right) n \bar{p}(n) + \ell^{-1} n \bar{p}(n/\ell^2) \equiv (1+\ell) n \bar{p}(n) \pmod{2^\gamma}. \quad (1.10)$$

2. PRELIMINARIES

Let λ be a nonnegative integer, N be a positive integer, and χ be a Dirichlet character modulo $4N$. A holomorphic function $f(\tau)$ on the complex upper half-plane \mathbb{H} is called a (weakly) holomorphic half-integral weight modular form with weight $\lambda + 1/2$ and character χ if it is holomorphic (resp. meromorphic) at the cusps and if

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d) \left(\frac{c}{d}\right)^{2\lambda+1} \epsilon_d^{-1-2\lambda} (c\tau+d)^{\lambda+1/2} f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N),$$

where

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

Suppose that f has the Fourier expansion $f(\tau) = \sum_n a(n) q^n$. For a prime ℓ and a Dirichlet character χ , define the weight $\lambda + 1/2$ Hecke operator $T_{\lambda+1/2}(\ell^2, \chi)$ (or simply $T_{\lambda+1/2}(\ell^2)$ when the character is trivial) by

$$f|_{T_{\lambda+1/2}(\ell^2, \chi)}(\tau) = \sum_n \left(a(\ell^2 n) + \chi^*(\ell) \left(\frac{n}{\ell}\right) \ell^{\lambda-1} a(n) + \chi^*(\ell^2) \ell^{2\lambda-1} a(n/\ell^2) \right) q^n, \quad (2.1)$$

where χ^* is the Dirichlet character given by $\chi^*(m) := \left(\frac{(-1)^\lambda}{m}\right) \chi(m)$ and $a(n/\ell^2) = 0$ if $\ell^2 \nmid n$.

A computation shows that the Hecke operator $T(\ell^2)$ commutes with the operator $\theta := q \frac{d}{dq}$ in the following way:

$$\ell^2 \theta \left(f|_{T_{\lambda+1/2}(\ell^2, \chi)} \right) = (\theta f)|_{T_{\lambda+2+1/2}(\ell^2, \chi)}. \quad (2.2)$$

We define the Fricke involution W_N on $M_k^1(\Gamma_0(N))$ by

$$F|_k W_N(\tau) := \left(-i\sqrt{N}\tau\right)^{-k} F(-1/N\tau). \quad (2.3)$$

Using the fact that $\eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau)$, we find that the functions $R(\tau)$ and $\overline{P}(\tau)$ are related by

$$R|_{-\frac{1}{2}} W_{16}(\tau) = \sqrt{-4i\tau} \left(\frac{\eta(-1/2\tau)}{\eta^2(-1/\tau)} \right) = \sqrt{8} \overline{P}(\tau). \quad (2.4)$$

We let $M_k(\Gamma)$ (resp. $M_k^!(\Gamma)$) denote the space of modular forms (resp. weakly holomorphic modular forms) of weight k for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. For $4 \leq k \in 2\mathbb{Z}$, $E_k(\tau)$ denotes the weight k Eisenstein series

$$E_k(\tau) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k(\mathrm{SL}_2(\mathbb{Z})),$$

where B_k is the k^{th} Bernoulli number and $\sigma_m(n)$ is the sum of divisors function $\sigma_m(n) := \sum_{d|n} d^m$. Some other useful modular forms are

$$E(\tau) := 2E_2(2\tau) - E_2(\tau) \in M_2(\Gamma_0(2)), \quad (2.5)$$

$$\psi(\tau) := \frac{\eta^{16}(2\tau)}{\eta^8(\tau)} = R^{-8}(\tau/8) \in M_4(\Gamma_0(2)), \quad (2.6)$$

$$\rho(\tau) := \frac{\eta^{16}(\tau)}{\eta^8(2\tau)} = 2^6 \psi|_4 W_2(\tau) = \overline{P}^{-8}(\tau) \in M_4(\Gamma_0(2)). \quad (2.7)$$

3. PROOF OF THEOREM 1.4

Let ℓ be an odd prime. To prove the statement

$$\overline{p}(\ell^2 n) + \left(\frac{-n}{\ell} \right) \ell^{-2} \overline{p}(n) + \ell^{-3} \overline{p}(n/\ell^2) \equiv (1 + \ell) \overline{p}(n) \pmod{16}$$

from Theorem 1.4, we will show that

$$\overline{P}|_{T_{-\frac{1}{2}}(\ell^2)}(\tau) - (1 + \ell) \overline{P}(\tau) \equiv 0 \pmod{16}. \quad (3.1)$$

To do this we need the congruence

$$\overline{P}(\tau) \equiv \overline{P}^{-7}(\tau) \pmod{16},$$

which follows immediately from the fact that $\rho(\tau) \equiv 1 \pmod{16}$ and $\rho(\tau) = \overline{P}^{-8}(\tau)$. We will also need the following proposition.

Proposition 3.1. *Let ℓ be an odd prime. Then $\overline{P}^{-7}(\tau)$ is an eigenform for the weight $7/2$ Hecke operator $T_{\frac{7}{2}}(\ell^2)$ with eigenvalue $\ell^5 + 1$.*

Let us assume for the moment that this proposition is true. Since $\ell^{-1} \equiv \ell^3 \pmod{16}$, the Hecke operators $T_{-\frac{1}{2}}(\ell^2)$ and $T_{\frac{7}{2}}(\ell^2)$ agree modulo 16, so we have

$$\begin{aligned} \overline{P}|_{T_{-\frac{1}{2}}(\ell^2)}(\tau) - (1 + \ell) \overline{P} &\equiv \overline{P}^{-7}|_{T_{\frac{7}{2}}(\ell^2)}(\tau) - (1 + \ell) \overline{P}^{-7} \\ &\equiv (\ell^5 + 1) \overline{P}^{-7} - (1 + \ell) \overline{P}^{-7} \\ &\equiv (\ell^5 - \ell) \overline{P}^{-7} \equiv 0 \pmod{16}, \end{aligned}$$

and the first statement in Theorem 1.4 follows.

Proof of Proposition 3.1. Recall that the Fricke involution W_N commutes with the Hecke operator $T(\ell^2)$ when $\ell \nmid N$ (see, for example, chapter 1 of [15]). Therefore $\overline{P}^{-7}(\tau)$ is an eigenform for $T(\ell^2)$ if and only if $R^{-7}(\tau)$ is an eigenform for $T(\ell^2)$. Since $R^{-7}(\tau) \in M_{\frac{7}{2}}(\Gamma_0(16))$, we use the dimension formulas of Cohen-Oesterle (see [6]) to compute $\dim \left(M_{\frac{7}{2}}(\Gamma_0(16)) \right) = 8$, and we explicitly compute that there exists a basis of the form $\{f_m(\tau)\}_{m=0}^7$ where $f_m(\tau) = q^m + O(q^8)$. Notice that $R^{-7}(\tau)$ vanishes to order 7, so $R^{-7}(\tau) = f_7(\tau)$. The form $R^{-7}(\tau) = q^7 + O(q^{15})$ is supported on exponents which are $7 \pmod{8}$ so the only coefficients which appear in the q -expansion of $R^{-7}|_{T_{\frac{7}{2}}(\ell^2)}$ are those for which the exponent is $7 \pmod{8}$. So we have

$$R^{-7}|_{T_{\frac{7}{2}}(\ell^2)}(\tau) = \lambda_{\ell} q^7 + O(q^{15}) \in M_{\frac{7}{2}}(\Gamma_0(16))$$

for some $\lambda_\ell \in \mathbb{Z}$ and therefore $R^{-7}|T_{\frac{7}{2}}(\ell^2) = \lambda_\ell f_7 = \lambda_\ell R^{-7}$. To determine the eigenvalue λ_ℓ we use (2.4) to compute

$$\begin{aligned}\overline{P}^{-7}|T_{\frac{7}{2}}(\ell^2) &= 8^{7/2}R^{-7}|_{\frac{7}{2}}W_{16}|T_{\frac{7}{2}}(\ell^2) \\ &= 8^{7/2}\lambda_\ell R^{-7}|_{\frac{7}{2}}W_{16} = \lambda_\ell \overline{P}^{-7},\end{aligned}$$

so λ_ℓ is the constant term of $\overline{P}^{-7}|T_{\frac{7}{2}}(\ell^2)$ which, by (2.1), is equal to $\ell^5 + 1$. \square

To prove the statement

$$\ell^2 n \overline{p}(\ell^2 n) + \left(\frac{-n}{\ell}\right) n \overline{p}(n) + \ell^{-1} n \overline{p}(n/\ell^2) \equiv (1 + \ell) n \overline{p}(n) \pmod{2^\gamma}$$

from Theorem 1.4, we will prove the equivalent statement

$$(\theta \overline{P})|T_{\frac{3}{2}}(\ell^2)(\tau) - (1 + \ell)\theta \overline{P}(\tau) \equiv 0 \pmod{2^\gamma}, \quad (3.2)$$

where we recall that

$$\gamma = \begin{cases} 5 & \text{if } \ell \equiv 3 \pmod{8} \\ 6 & \text{if } \ell \equiv 5, 7 \pmod{8} \\ 7 & \text{if } \ell \equiv 1 \pmod{8}. \end{cases}$$

To do this we will proceed as before, replacing $\theta \overline{P}(\tau)$ by $\theta(\overline{P}^{-31}(\tau) + 64\overline{P}(\tau)\psi(\tau))$, a switch that is justified by the following lemma.

Lemma 3.2.

$$\theta \overline{P} \equiv \theta(\overline{P}^{-31} + 64\overline{P}\psi) \pmod{128}. \quad (3.3)$$

Proof. Since $\rho(\tau) = \overline{P}^{-8}(\tau)$, an equivalent form of (3.3) is

$$\theta(\overline{P}(1 - \rho^4 - 64\psi)) \equiv 0 \pmod{128}. \quad (3.4)$$

To prove (3.4) we will need the derivatives

$$\theta \rho = -\frac{1}{3}\rho(E - E_2), \quad (3.5)$$

$$\theta \psi = \frac{1}{3}\psi(2E + E_2). \quad (3.6)$$

Since $\rho^4 \equiv 1 \pmod{64}$ and $\overline{P} \equiv 1 \pmod{2}$ we have

$$\begin{aligned}\theta(\overline{P}(1 - \rho^4 - 64\psi)) &= (\theta \overline{P})(1 - \rho^4 - 64\psi) + \overline{P}\theta(1 - \rho^4 - 64\psi) \\ &\equiv \overline{P} \left(\frac{4}{3}\rho^4(E - E_2) - \frac{64}{3}\psi(2E + E_2) \right) \pmod{128}.\end{aligned}$$

Since $2E + E_2 \equiv 1 \pmod{2}$, it remains to show that

$$E - E_2 \equiv 16\psi \pmod{32}. \quad (3.7)$$

Computation shows that

$$16\psi(\tau) = \frac{16}{240}(E_4(\tau) - E_4(2\tau)) = 16 \sum_{n=1}^{\infty} (\sigma_3(n) - \sigma_3(n/2)) q^n.$$

We also have

$$E(\tau) - E_2(\tau) = 2(E_2(2\tau) - E_2(\tau)) = 48 \sum_{n=1}^{\infty} (\sigma_1(n) - \sigma_1(n/2)) q^n.$$

Congruence (3.7) follows since $\sigma_3(n) \equiv \sigma_1(n) \pmod{2}$. \square

We return now to the proof of (3.2); in light of (2.2) and (3.3), it is enough to prove the statements

$$\theta(\ell^2 \overline{P}^{-31}|T_{-\frac{1}{2}}(\ell^2) - (1 + \ell)\overline{P}^{-31}) \equiv 0 \pmod{2^\gamma}, \quad (3.8)$$

$$\theta(\ell^2 \overline{P}\psi|T_{-\frac{1}{2}}(\ell^2) - (1 + \ell)\overline{P}\psi) \equiv 0 \pmod{2}. \quad (3.9)$$

We would like to exchange weight $-1/2$ for weight $31/2$ in the first case and weight $7/2$ in the second case. The exchange is trivial in (3.9). To see that it is justified in (3.8), let $\bar{P}^{-31}(\tau) = \sum_{n=0}^{\infty} a(n)q^n$. Since $\bar{P}(\tau) \equiv 1 \pmod{2}$, $a(n)$ is even for all $n \geq 1$. Hence

$$\ell^2 \bar{P}^{-31} | T_{\frac{31}{2}}(\ell^2)(\tau) - \ell^2 \bar{P}^{-31} | T_{-\frac{1}{2}}(\ell^2)(\tau) \equiv \sum_{n=1}^{\infty} \left(\frac{-n}{\ell} \right) (\ell^{16} - 1) a(n) q^n \equiv 0 \pmod{128},$$

since $\ell^{16} \equiv 1 \pmod{64}$. Therefore (3.8) is implied by

$$\theta \left(\ell^2 \bar{P}^{-31} | T_{\frac{31}{2}}(\ell^2)(\tau) - (1 + \ell) \bar{P}^{-31}(\tau) \right) \equiv 0 \pmod{2^7}. \quad (3.10)$$

The truth of (3.10) will follow from the next proposition.

Proposition 3.3. *The function $\bar{P}^{-31}(\tau)$ is an eigenform for the Hecke operator $T_{\frac{31}{2}}(\ell^2)$ modulo 2^{12} with eigenvalue $\ell^{29} + 1$.*

Assuming for the moment that this proposition is true, we have

$$\ell^2 \bar{P}^{-31} | T_{\frac{31}{2}}(\ell^2)(\tau) - (1 + \ell) \bar{P}^{-31}(\tau) \equiv (\ell^2(\ell^{29} + 1) - (1 + \ell)) \bar{P}^{-31}(\tau) \pmod{2^{12}}.$$

Then (3.10) follows from the congruence

$$\ell^{31} + \ell^2 - \ell - 1 \equiv 0 \pmod{2^{\gamma-1}},$$

which is easily checked.

Proof of Proposition 3.3. Since the form $R^{-1}(\tau)$ is supported on exponents which are $7 \pmod{8}$ and the form $R(\tau)$ is supported on exponents which are $1 \pmod{8}$, we conclude that

$$\tilde{R}_{\ell}(\tau) := R^{-31} | T_{\frac{31}{2}}(\ell^2)(\tau) \cdot R^7(\tau) \in M_{12}^!(\Gamma_0(16))$$

is supported on exponents divisible by 8. By [4, Lemma 7], we have $\tilde{R}_{\ell}(\tau/8) \in M_{12}^!(\Gamma_0(2))$, and a computation involving the Fricke involution W_2 shows that $\tilde{R}_{\ell}(\tau/8)$ is holomorphic at the cusp 0. So $\tilde{R}_{\ell}(\tau/8) \in M_{12}(\Gamma_0(2))$, which implies that there exist integers c_j , $0 \leq j \leq 3$, such that

$$\tilde{R}_{\ell}(\tau/8) = c_0 \rho^3(\tau) + c_1 \rho^2(\tau) \psi(\tau) + c_2 \rho(\tau) \psi^2(\tau) + c_3 \psi^3(\tau).$$

Since $\psi(8\tau) = R^{-8}(\tau)$, we have

$$R^{-31} | T_{\frac{31}{2}}(\ell^2)(\tau) = c_0 R^{-7}(\tau) \rho^3(8\tau) + c_1 R^{-15}(\tau) \rho^2(8\tau) + c_2 R^{-23}(\tau) \rho(8\tau) + c_3 R^{-31}(\tau). \quad (3.11)$$

Applying the Fricke involution W_{16} to (3.11) and multiplying by $8^{31/2}$, we obtain

$$\bar{P}^{-31} | T_{\frac{31}{2}}(\ell^2) = 2^{36} c_0 \bar{P}^{-7} \psi^3 + 2^{24} c_1 \bar{P}^{-15} \psi^2 + 2^{12} c_2 \bar{P}^{-23} \psi + c_3 \bar{P}^{-31}(\tau).$$

Using the fact that ψ vanishes at ∞ and recalling (2.1), we conclude that $c_3 = \ell^{29} + 1$. Hence,

$$\bar{P}^{-31} | T_{\frac{31}{2}}(\ell^2) \equiv (\ell^{29} + 1) \bar{P}^{-31}(\tau) \pmod{2^{12}},$$

which completes the proof of Proposition 3.3. \square

It remains only to prove (3.9) to finish the proof of Theorem 1.4. Since $\psi(\tau) \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod{2}$, we only need to show that

$$\left(\sum_{n=0}^{\infty} q^{(2n+1)^2} \right) | T_{\frac{7}{2}}(\ell^2) \equiv 0 \pmod{2}.$$

Let $\sum_{n=0}^{\infty} q^{(2n+1)^2} = \sum_{n=1}^{\infty} b(n) q^n$. Then

$$b(\ell^2 n) + \left(\frac{-n}{\ell} \right) b(n) + b(n/\ell^2) \equiv \begin{cases} b(\ell^2 n) + b(n) \equiv 1 + 1 \equiv 0 \pmod{2} & \text{if } n = m^2, m \text{ odd}, (m, \ell) = 1 \\ b(\ell^2 n) + b(m^2) \equiv 1 + 1 \equiv 0 \pmod{2} & \text{if } n = \ell^2 m^2, m \text{ odd} \\ b(\ell^2 n) + \left(\frac{-n}{\ell} \right) b(n) \equiv 0 + 0 \equiv 0 \pmod{2} & \text{if } n \neq \text{odd square.} \end{cases}$$

Congruence (3.9) follows, and Theorem 1.4 is proved. \square

4. PROOF OF THEOREMS 1.2 AND 1.3 MODULO 3 AND 5

In this section we prove the congruences (1.5) and (1.8) for the moduli 3 and 5. The proofs use similar techniques, so we present them together. Let ℓ be an odd prime and define $t_\ell := 15/(15, \ell)$.

Recalling that $\overline{S}(\tau) = \sum_n \overline{\text{spt}}(n)q^n$, we find that Theorem 1.2 for the moduli $t = 3, 5$ is equivalent to

$$\overline{S}_\ell(\tau) := \overline{S}|T_{\frac{3}{2}}(\ell^2)(\tau) - (1 + \ell)\overline{S}(\tau) \equiv 0 \pmod{t_\ell}. \quad (4.1)$$

We will need the following lemma.

Lemma 4.1. $4E_2(2\tau) - E_2(\tau) \equiv 3E(\tau)\rho(\tau) \pmod{45}$.

Proof. Since $E(\tau)\rho(\tau) \in M_6(\Gamma_0(2))$, it is a linear combination of $E_6(\tau)$ and $E_6(2\tau)$. We compute that

$$\begin{aligned} 3E(\tau)\rho(\tau) &= \frac{1}{21} (64E_6(2\tau) - E_6(\tau)) = 3 - 24 \sum_{n=1}^{\infty} (64\sigma_5(n/2) - \sigma_5(n)) q^n \\ &\equiv 4E_2(2\tau) - E_2(\tau) \pmod{45}. \end{aligned} \quad \square$$

We apply Lemma 4.1 to equation (1.4) to obtain

$$\overline{M}(\tau) - \overline{S}(\tau) \equiv -\frac{1}{4}\overline{P}(\tau) \cdot E(\tau)\rho(\tau) \equiv 11\overline{P}^{-7}(\tau)E(\tau) \pmod{15}.$$

We omit the proof of the next proposition since it is similar to the proof of Proposition 3.1.

Proposition 4.2. *Let ℓ be an odd prime. Then $E(\tau)\overline{P}^{-7}(\tau)$ is an eigenform for $T_{\frac{11}{2}}(\ell^2)$ with eigenvalue $\ell^9 + 1$.*

Since $\ell^4 \equiv 1 \pmod{t_\ell}$, the Hecke operators $T_{\frac{3}{2}}(\ell^2)$ and $T_{\frac{11}{2}}(\ell^2)$ agree modulo t_ℓ . The function \overline{M} is an eigenform for $T_{\frac{3}{2}}(\ell^2)$, therefore

$$\overline{M}|T_{\frac{11}{2}}(\ell^2) \equiv (1 + \ell)\overline{M} \pmod{t_\ell},$$

from which we can conclude

$$\overline{S}_\ell \equiv - (11(\ell^9 + 1)\overline{P}^{-7}E - 11(\ell + 1)\overline{P}^{-7}E) \equiv 4\ell(\ell^8 - 1)\overline{P}^{-7}(\tau)E(\tau) \equiv 0 \pmod{t_\ell}.$$

This completes the proof of (4.1).

To prove Theorem 1.3 for the moduli $t = 3, 5$ we will prove that

$$S2_\ell(\tau) := S2|T_{\frac{3}{2}}(\ell^2)(\tau) - (1 + \ell)S2(\tau) \equiv 0 \pmod{t_\ell}, \quad (4.2)$$

where we recall that $S2(\tau) = \sum_n (-1)^n M2\text{spt}(n)q^{8n-1}$. We omit the proof of the next lemma since it is similar to the proof of Lemma 4.1.

Lemma 4.3. $E_2(2\tau) - E_2(\tau) \equiv 24E(\tau)\psi(\tau) \pmod{45}$.

We apply Lemma 4.3 to (1.7) to obtain

$$M2(\tau) - S2(\tau) \equiv R(\tau) \cdot E(8\tau)\psi(8\tau) \equiv R(\tau)^{-7}E(8\tau) \pmod{15}.$$

Since $R(\tau)^{-7}E(8\tau)|_{\frac{11}{2}}W_{16} = 2^{-27/2}E(\tau)\overline{P}^{-7}(\tau)$, and since W_{16} commutes with $T(\ell^2)$ we have, by Proposition 4.2, that $E(8\tau)R(\tau)^{-7}$ is an eigenform for $T_{\frac{11}{2}}(\ell^2)$ with eigenvalue $\ell^9 + 1$. As in the case of \overline{M} , we are justified in switching the weight from $\frac{3}{2}$ to $\frac{11}{2}$, and since $M2$ is an eigenform for $T_{\frac{3}{2}}(\ell^2)$, we have

$$M2|T_{\frac{11}{2}}(\ell^2) \equiv (1 + \ell)\overline{M} \pmod{t_\ell},$$

from which we can conclude

$$S2_\ell(\tau) \equiv - [(\ell^9 + 1)R(\tau)^{-7}E(8\tau) - (\ell + 1)R(\tau)^{-7}E(8\tau)] \equiv -\ell(\ell^8 - 1)R(\tau)^{-7}E(8\tau) \equiv 0 \pmod{t_\ell}.$$

This completes the proof of (4.1). \square

5. PROOF OF THEOREM 1.2 MODULO POWERS OF 2

Recalling the definition of \overline{S}_ℓ in (4.1), we must prove that

$$\overline{S}_\ell(\tau) \equiv 0 \pmod{2^\alpha},$$

where

$$\alpha = \begin{cases} 6 & \text{if } \ell \equiv 3 \pmod{8} \\ 7 & \text{if } \ell \equiv 5, 7 \pmod{8} \\ 8 & \text{if } \ell \equiv 1 \pmod{8}. \end{cases}$$

By (1.4) and (3.5) we obtain

$$\overline{M}(\tau) - \overline{S}(\tau) = 2\theta\overline{P}(\tau) - \frac{1}{4}\overline{h}(\tau), \quad (5.1)$$

where $\overline{h}(\tau)$ is the weight $3/2$ weakly holomorphic modular form

$$\overline{h}(\tau) := E(\tau) \frac{\eta(2\tau)}{\eta^2(\tau)} \in M_{\frac{3}{2}}^!(\Gamma_0(16)).$$

Using (2.2) and the fact that $\overline{M}|T_{\frac{3}{2}}(\ell^2)(\tau) = (1 + \ell)\overline{M}(\tau)$ we obtain

$$\overline{S}_{\ell}(\tau) = \frac{1}{4} \left(h|T_{\frac{3}{2}}(\ell^2)(\tau) - (1 + \ell)\overline{h}(\tau) \right) - 2\theta \left(\ell^2\overline{P}|T_{-\frac{1}{2}}(\ell^2)(\tau) - (1 + \ell)\overline{P}(\tau) \right), \quad (5.2)$$

so to prove (1.5) it is enough to prove the statements

$$\overline{h}|T_{\frac{3}{2}}(\ell^2)(\tau) - (1 + \ell)\overline{h}(\tau) \equiv 0 \pmod{256}, \quad (5.3)$$

$$\theta \left(\ell^2\overline{P}|T_{-\frac{1}{2}}(\ell^2) - (1 + \ell)\overline{P}(\tau) \right) \equiv 0 \pmod{2^{\alpha-1}}. \quad (5.4)$$

The second statement has been established in (3.2), and the first statement follows from the next proposition, which completes the proof of Theorem 1.2.

Proposition 5.1.

$$\overline{h}|T_{\frac{3}{2}}(\ell^2)(\tau) - (1 + \ell)\overline{h}(\tau) \equiv 0 \pmod{2^{12}}. \quad (5.5)$$

Proof. Define $\overline{g}(\tau) \in M_{\frac{3}{2}}^!(\Gamma_0(16))$ by

$$\overline{g}(\tau) := \frac{1}{\sqrt{8}} \overline{h}(\tau)|_{\frac{3}{2}} W_{16} = E(8\tau) \frac{\eta(8\tau)}{\eta^2(16\tau)}.$$

From [2, Corollary 4], we have

$$\overline{g}|T_{\frac{3}{2}}(\ell^2)(\tau) - \overline{g}(\tau) = \ell f_{\ell^2}(\tau)$$

where $f_{\ell^2}(\tau)$ is given by

$$f_{\ell^2}(\tau) = \left(1 + \sum_{n=1}^{(\ell^2-1)/8} c_n j_2^n(8\tau) \right) \overline{g}(\tau)$$

for some $c_n \in \mathbb{Z}$, and $j_2(\tau)$ is the Hauptmodul on $\Gamma_0(2)$ given by

$$j_2(\tau) := \left(\frac{\eta(\tau)}{\eta(2\tau)} \right)^{24} \in M_0^!(\Gamma_0(2)).$$

Applying the Fricke involution W_{16} and multiplying by $\sqrt{8}$, we obtain

$$\overline{h}(\tau)|T_{\frac{3}{2}}(\ell^2) - \overline{h}(\tau) = \ell \left(1 + \sum_{n=1}^{s_{\ell}} c_n j_2^n(-1/2\tau) \right) \overline{h}(\tau).$$

Since $j_2(-1/2\tau) = 2^{12} j_2^{-1}(\tau)$, we conclude that

$$\overline{h}(\tau)|T_{\frac{3}{2}}(\ell^2) - \overline{h}(\tau) \equiv \ell \overline{h}(\tau) \pmod{2^{12}},$$

from which (5.5) follows. □

6. PROOF OF THEOREM 1.3 MODULO POWERS OF 2

Recalling the definition of $S2_{\ell}$ in (4.2), we must prove that

$$S2_{\ell}(\tau) \equiv 0 \pmod{2^{\beta}}, \quad (6.1)$$

where

$$\beta := \begin{cases} 1 & \text{if } \ell \equiv 3 \pmod{8} \\ 2 & \text{if } \ell \equiv 5 \pmod{8} \\ 3 & \text{if } \ell \equiv 1, 7 \pmod{8}. \end{cases}$$

We will need the following lemma.

Lemma 6.1. $E_2(2\tau) - E_2(\tau) \equiv 24\psi(\tau) - 16\psi^2(\tau) + 32\psi^4(\tau) \pmod{64}$.

Proof. If $m \in \mathbb{Z}$ then $E_2(\tau) - mE_2(m\tau) \in M_2(\Gamma_0(m))$. Therefore

$$g(\tau) := E_2(2\tau) - 64E_2(128\tau) + 64E_2(64\tau) - E_2(\tau) \in M_2(\Gamma_0(128)),$$

and $g(\tau) \equiv E_2(2\tau) - E_2(\tau) \pmod{64}$. We compute that

$$g(\tau) \equiv \psi(\tau) - 16\psi^2(\tau) + 32\psi^4(\tau) \pmod{64}$$

by computing sufficiently many terms of

$$\tilde{g}(\tau) := g(\tau)E(\tau)\rho^3(\tau) - 24\psi(\tau)\rho^3(\tau) + 16\psi^2(\tau)\rho^2(\tau) - 32\psi^4(\tau) \in M_{16}(\Gamma_0(128))$$

to see that $\tilde{g}(\tau) \equiv 0 \pmod{64}$. Since $\tilde{g}(\tau) \equiv g(\tau) - 24\psi(\tau) + 16\psi^2(\tau) - 32\psi^4(\tau) \pmod{64}$, this completes the proof. \square

We apply Lemma 6.1 to (1.7) along with the fact that $\psi(8\tau) = R^{-8}(\tau)$ to obtain

$$M2 - S2 \equiv R^{-7} - 2R^{-15} + 4R^{-31} \pmod{8}.$$

Let ℓ be an odd prime. By Proposition 3.1, we have $R^{-7}|T_{\frac{7}{2}}(\ell^2) = (\ell^5 + 1)R^{-7}$. Therefore

$$-S2_\ell \equiv 2 \left(R^{-15}|T_{\frac{15}{2}}(\ell^2) - (1 + \ell)R^{-15} \right) + 4 \left(R^{-31}|T_{\frac{31}{2}}(\ell^2) - (1 + \ell)R^{-31} \right) \pmod{8}.$$

This proves Theorem 1.3 when $\ell \equiv 3 \pmod{8}$. Suppose now that $\ell \not\equiv 3 \pmod{8}$. To simplify notation, let $\rho_8 = \rho(8\tau)$. Recall (3.11) and the discussion that follows it, which together imply that for some integers c_0, c_1, c_2 depending on ℓ , we have

$$R^{-31}|T_{\frac{31}{2}}(\ell^2) = c_0R^{-7}\rho_8^3 + c_1R^{-15}\rho_8^2 + c_2R^{-23}\rho_8 + (\ell^{29} + 1)R^{-31}. \quad (6.2)$$

Similarly, it can be shown that there exists some $d_0 \in \mathbb{Z}$ depending on ℓ such that

$$R^{-15}|T_{\frac{15}{2}}(\ell^2)(\tau) = d_0R^{-7}\rho_8 + (\ell^{13} + 1)R^{-15}, \quad (6.3)$$

and since $\rho \equiv 1 \pmod{8}$, we have

$$\begin{aligned} -S2_\ell &\equiv 2 \left(d_0R^{-7}\rho_8 + (\ell^{13} - \ell)R^{-15} \right) + 4 \left(c_0R^{-7}\rho_8^3 + c_1R^{-15}\rho_8^2 + c_2R^{-23}\rho_8 + (\ell^{29} - \ell)R^{-31} \right) \\ &\equiv 2d_0R^{-7} + 4 \left(c_0R^{-7} + c_1R^{-15} + c_2R^{-23} \right) \pmod{8}. \end{aligned}$$

The following proposition completes the proof.

Proposition 6.2. *Let c_0, c_1, c_2, d_0 be as above. Then the following are true:*

- (1) *If $\ell \equiv 1 \pmod{4}$ then $d_0 \equiv 0 \pmod{4}$.*
- (2) *If $\ell \equiv 1 \pmod{8}$ then $c_0 \equiv 0 \pmod{2}$.*
- (3) *If $\ell \equiv 7 \pmod{8}$ then $d_0 \equiv 2c_0 \pmod{4}$.*
- (4) *If $\ell \equiv 1, 7 \pmod{8}$ then $c_1 \equiv c_2 \equiv 0 \pmod{2}$.*

Proof. By (2.1), (6.2), (6.3), and the fact that $R^{-7}\rho_8^3 = q^7 - 41q^{15} + 789q^{23} + O(q^{31})$ and $R^{-15}\rho_8^2 = q^{15} - 17q^{23} + O(q^{31})$, the coefficients d_0, c_0, c_1, c_2 are given by

$$d_0 = a_{15}(7\ell^2), \quad c_0 = a_{31}(7\ell^2), \quad c_1 = a_{31}(15\ell^2) + 41c_0, \quad c_2 = a_{31}(23\ell^2) + 17c_1 - 789c_0,$$

where

$$\sum_{n=1}^{\infty} a_r(n)q^n := R^{-r}(\tau) = \left(\sum_{n=0}^{\infty} q^{(2n+1)^2} \right)^r.$$

For $f(\tau) = \sum_{n=1}^{\infty} a(n)q^n \in M_{k+1/2}(\Gamma_0(4N))$ we define the t^{th} Shimura lift \mathcal{S}_t by

$$\mathcal{S}_t f(\tau) = \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \left(\frac{(-1)^{k-1} 4t}{d} \right) a(|t|n^2/d^2) \right) q^n. \quad (6.4)$$

If $f \in S_{k+1/2}(\Gamma_0(4N))$, then $\mathcal{S}_t f \in S_{2k}(\Gamma_0(2N))$ (see, for instance, [15, Section 3.3]). Now, $\rho\psi$ is a cusp form since ψ vanishes at ∞ and ρ vanishes at 0. We compute $\mathcal{S}_t \rho_8 R^{-r}$, which is a cusp form. Since $\rho \equiv 1 \pmod{4}$ we have

$$\mathcal{S}_t \rho_8 R^{-r} \equiv \mathcal{S}_t R^{-r} \pmod{4}.$$

We will now compute the lifts \mathcal{S}_t and write the resulting functions in terms of F and ϑ_0 , where

$$F := \sum_{n=0}^{\infty} \sigma_1(2n+1)q^{2n+1} \in M_2(\Gamma_0(4)), \quad \vartheta_0 := \sum_{n=-\infty}^{\infty} q^{n^2} \in M_{\frac{1}{2}}(\Gamma_0(4)).$$

(1) Suppose $\ell \equiv 1 \pmod{4}$. We compute sufficiently many coefficients to determine that

$$\begin{aligned} \mathcal{S}_7 \rho_8 R^{-15} &= -1287 F^3 \vartheta_0^{32} + 2^6 \cdot 1287 F^4 \vartheta_0^{28} - 2^8 \cdot 6721 F^5 \vartheta_0^{24} + 2^{12} \cdot 2145 F^6 \vartheta_0^{20} \\ &\quad + 2^{16} \cdot 1859 F^7 \vartheta_0^{16} - 2^{20} \cdot 1287 F^8 \vartheta_0^{12} + 2^{24} \cdot 143 F^9 \vartheta_0^8. \end{aligned}$$

Since $\vartheta_0(\tau) \equiv 1 \pmod{2}$, we conclude that

$$\mathcal{S}_7 R^{-15} \equiv \mathcal{S}_7 \rho_8 R^{-15} \equiv F^3 \pmod{4}.$$

Now, $a_{15}(7\ell^2)$ is the coefficient of q^ℓ in $\mathcal{S}_7 R^{-15}$. We must show that the form $F^3 \pmod{4}$ is supported on exponents which are $3 \pmod{4}$. Let $\chi_{-4} := \left(\frac{-4}{\bullet}\right)$, and let $F_{\chi_{-4}}$ denote the twist of F by χ_{-4} . Then

$$\tilde{F} := \frac{1}{2} (F^3 + F_{\chi_{-4}}^3) \in M_2(\Gamma_0(16)),$$

and by computing sufficiently many coefficients we find that $\tilde{F} \equiv 0 \pmod{4}$.

(2) Suppose $\ell \equiv 1 \pmod{8}$. We compute that

$$\begin{aligned} \mathcal{S}_7 \rho_8 R^{-31} &= -693 F^3 \vartheta_0^{64} + 2^7 \cdot 693 F^4 \vartheta_0^{60} - 14158837 F^5 \vartheta_0^{56} + 2^4 \cdot 74274739 F^6 \vartheta_0^{52} \\ &\quad - 45253573295 F^7 \vartheta_0^{48} + 2^5 \cdot 20433347725 F^8 \vartheta_0^{44} + 2^8 \cdot 29560308687 F^9 \vartheta_0^{40} \\ &\quad - 2^{14} \cdot 28133747817 F^{10} \vartheta_0^{36} + 2^{15} \cdot 250545162231 F^{11} \vartheta_0^{32} - 2^{23} \cdot 9410428671 F^{12} \vartheta_0^{28} \\ &\quad + 2^{23} \cdot 53378995173 F^{13} \vartheta_0^{24} - 2^{27} \cdot 11400290027 F^{14} \vartheta_0^{20} + 2^{32} \cdot 697257169 F^{15} \vartheta_0^{16} \\ &\quad - 2^{35} \cdot 43328593 F^{16} \vartheta_0^{12} - 2^{40} \cdot 122815 F^{17} \vartheta_0^8, \end{aligned} \quad (6.5)$$

and therefore

$$\mathcal{S}_7 R^{-31} \equiv F^3 + F^5 + F^7 \pmod{2}.$$

Notice that $F \equiv \sum_{n \geq 0} q^{(2n+1)^2} \pmod{2}$, so the the form $F^3 + F^5 + F^7 \pmod{2}$ is supported only on exponents which are $3, 5, 7 \pmod{8}$. The quantity $c_0 = a_{31}(7\ell^2)$ is the coefficient of q^ℓ in $\mathcal{S}_7 R^{-31}$, therefore $c_0 \equiv 0 \pmod{2}$.

(3) Suppose $\ell \equiv 7 \pmod{8}$. From (6.5) we have

$$\mathcal{S}_7 R^{-31} \equiv 3F^3 + 3F^5 + F^7 \pmod{4}.$$

The quantity $d_0 - 2c_0 = a_{15}(7\ell^2) - 2a_{31}(7\ell^2)$ is the coefficient of q^ℓ in

$$\mathcal{S}_7 R^{-15} - 2 \mathcal{S}_7 R^{-31} \equiv 3F^3 + 2F^5 + 2F^7 \pmod{4}.$$

Since $F \equiv \sum_{n \geq 0} q^{(2n+1)^2} \pmod{2}$, the form $2F^5 \pmod{4}$ is supported on exponents which are $5 \pmod{8}$. We must show that the form $3F^3 + 2F^7 \pmod{4}$ is supported on exponents which are 1 or $3 \pmod{8}$. Then, since $\ell \equiv 7 \pmod{8}$, it will follow that $d_0 - 2c_0 \equiv 0 \pmod{4}$.

Define

$$f := 3F^3 \vartheta_0^{16} + 2F^7 \in M_{14}(\Gamma_0(4)).$$

Then $f \equiv 3F^3 + 2F^7 \pmod{4}$. Let $\chi_{-8} := \left(\frac{-8}{\bullet}\right)$, and let $f_{\chi_{-8}}$ denote the twist of f by χ_{-8} . Then

$$\tilde{f} := \frac{1}{2} (f - f_{\chi_{-8}}) \in M_{14}(\Gamma_0(64)),$$

and by computing sufficiently many coefficients we find that $\tilde{f} \equiv 0 \pmod{4}$.

(4) Suppose $\ell \equiv 1, 7 \pmod{8}$. To show that c_1 and c_2 are even, we will show

$$a_{31}(15\ell^2) + a_{31}(7\ell^2) \equiv 0 \pmod{2}, \quad (6.6)$$

$$a_{31}(23\ell^2) + a_{31}(15\ell^2) \equiv 0 \pmod{2}. \quad (6.7)$$

The left-hand side of (6.6) is the coefficient of q^ℓ in $\mathcal{S}_{15}R^{-31} + \mathcal{S}_7R^{-31}$. As before, we compute that

$$\mathcal{S}_{15}R^{-31} + \mathcal{S}_7R^{-31} \equiv F^5 \pmod{2}.$$

Since $\ell \equiv 1, 7 \pmod{8}$ and the form $F^5 \pmod{2}$ is supported on exponents which are $5 \pmod{8}$, congruence (6.6) is true. Similarly, the left-hand side of (6.7) is the coefficient of q^ℓ in $\mathcal{S}_{23}R^{-31} + \mathcal{S}_{15}R^{-31}$. We compute that

$$\mathcal{S}_{23}R^{-31} + \mathcal{S}_{15}R^{-31} \equiv F^3 \pmod{2}.$$

Since $\ell \equiv 1, 7 \pmod{8}$ and the form $F^3 \pmod{2}$ is supported only on exponents which are $3 \pmod{8}$, congruence (6.7) is true. \square

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